Received: *November 25, 1999*, Accepted: *December 17, 1999*

The thermodynamic potentials of Kerr-AdS black holes and their CFT duals

Karl Landsteiner and Esperanza Lopez

Theory Division CERN
CH-1211 Geneva 23, Switzerland
E-mail: Karl.Landsteiner@cern.ch, Esperanza.Lopez@cern.ch

Abstract: String or M-theory in the background of Kerr-AdS black holes is thought to be dual to the large N limit of certain conformal field theories on a rotating sphere at finite temperature. The five dimensional black hole is associated to $\mathcal{N}=4$ supersymmetric Yang-Mills theory on a rotating three-sphere and the four dimensional one to the superconformal field theory of coinciding M2 branes on a rotating two-sphere. The thermodynamic potentials can be expanded in inverse powers of the radius of the sphere. We compute the leading and subleading terms of this expansion in the field theory at one loop and compare them to the corresponding supergravity expressions. The ratios between these terms at weak and strong coupling turn out not to depend on the rotation parameters in the case of $\mathcal{N}=4$ SYM. For the field theory living on one M2 brane we find a subleading logarithmic term. No such term arises from the supergravity calculation.

Keywords: D-branes, Black Holes in String Theory.

JHEP12(1999)020

Contents

1. Introduction

The AdS/CFT correspondence relates string or M-theory on $M_n \times X^{D-n}$, where M_n is an space of negative curvature and X^{D-n} an Einstein manifold, to field theories living on the (conformal) boundary of these spaces [1, 2, 3] (see [4] for a comprehensive review). It is particularly interesting to consider M_n being asymptotically AdS black holes. The dual field theories are then at finite temperature, with the field theory temperature given by the Hawking temperature of the black hole.

An example is the AdS black hole with planar horizon. It arises as the near horizon limit of the near extremal D3-brane [5]. The dual field theory is $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. It turns out that the entropy, as calculated from the supergravity approximation to string theory, differs by a factor of 3/4 from the one calculated in the gauge theory [6]. This difference comes from the fact that the supergravity side corresponds to the gauge theory at infinite 't Hooft coupling, $\lambda = g_{YM}^2 N$, whereas the gauge theory calculation applies for weak 't Hooft coupling.[1,2] The coupling constant dependence of the entropy has been investigated in the strong [9] and weak [10] coupling limits. Because of the underlying conformal symmetry the temperature is the only scale in these problems. For this reason it is guaranteed that the entropy of the AdS black hole scales with the temperature in the same way as the conformal field theory. It seems therefore interesting to investigate situations where there are dimensionless parameters.

[1] In [7] it was argued on general grounds that there should be a factor of order one between the strong and weak coupling result in $\mathcal{N} = 4$ Yang-Mills.

[2] It is interesting to note that a factor of 4/5 between the thermal pressure at strong and weak coupling appears in the large N limit of an $O(N)$ scalar field theory in three dimensions [8].

This can be achieved by considering the conformal field theory living on a sphere. The entropy can then be multiplied by an a priori arbitrary function of the product of the temperature and the radius of the sphere. The supergravity duals are AdS black holes with spherical horizons. The precise dependence on the dimensionless parameter $\epsilon = 1/TR$ is easy to calculate in gravity. On the conformal field theory side it is however far from trivial to extract this dependence in a closed form. Expressions for the one loop energy of conformal fields on the three-sphere at finite temperature have been obtained long ago in [11] in form of infinite sums. One can however reside to a high temperature or large radius expansion

$$F = -VN^2T^4 \sum_{n=0}^{\infty} b_n(\lambda)\epsilon^n \,. \tag{1.1}$$

The leading term in such an expansion coincides with the flat space result. A strong/weak-coupling comparison of the subleading terms has been performed in [12]. The coupling constant dependence on the supergravity limit has been investigated in [13].

A way to include more dimensionless parameters is to consider Kerr-AdS black holes [14, 15, 16, 17]. The angular velocities give rise to new dimensionless parameters. The thermodynamic potential of these black holes presents a characteristic divergence when the angular velocity reaches the speed of light. The AdS correspondence relates these black holes to conformal field theories on a rotating Einstein universe. The thermodynamic potential of the conformal field theory shows the same divergence as its supergravity dual. In [18] it was shown that the ratio of the potentials in the extreme high temperature limit is independent of the angular velocities and still 3/4. A numerical calculation of the ratio between the free energies of the AdS black hole and the conformal field theories with varying rotation parameter has been performed in [16]. They found that the ratio depends on the rotation parameters although it always tends to 3/4 in the high temperature limit.

The aim of this paper is to calculate the subleading terms in the high temperature expansion of the thermodynamic potential in field theory with rotation parameters present. For the case of $\mathcal{N} = 4$ SYM on S^3 we have two independent angular velocities corresponding to the Cartan subalgebra of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ isometries of S^3. We take the generic case with both angular velocities different from zero and also different from each other. We find that the functional dependence of the subleading term is the same at weak and at strong coupling. Their ratio is given by $b_2(\infty)/b_2(0) = 3/2$, which coincides with the result in the non-rotating case [12]. This is remarkable since there does not seem to exist an obvious symmetry that constrains the functional form of this coefficient. Our result is also consistent with [16] since the ratio of the thermodynamic potentials at strong and weak coupling does depend on the angular velocities.

Four dimensional Kerr-AdS black holes are dual to the superconformal field theory arising on the world volume of M2 branes on a rotating two-sphere. We do not have an explicit formulation of this theory when more that one M2-brane coincide. We know however the theory for a single M2-brane, it is the $\mathcal{N}=8$ supersingleton theory. M-theory on $AdS_4 \times S^7$ has only one expansion parameter, the number of coincident M2-branes. It seems therefore the most natural analogue to the strong/weak coupling comparisons in four dimensions to compare the theories in three dimensions at $N=\infty$ and at $N=1$. We find that the subleading term at $N=1$ behaves logarithmically with the temperature. This is in strong contrast to the behaviour at $N=\infty$ where we find only polynomial behaviour in the high temperature expansion.

In order to derive subleading contributions to the thermodynamic potential of the field theory living on spheres, we have to evaluate sums over modes. We present a simple method to approximate sums to arbitrary accuracy when an small parameter is present. In our case the small parameter is $1/TR$. Our method might prove useful in other examples.

2. Supergravity calculation

2.1 Summary of thermodynamics of five dimensional Kerr-AdS black holes

The five dimensional Kerr-AdS metric has been derived and studied in [14]. We quote briefly some of their results. The metric can be written as

$$\begin{aligned} ds^2 = &-\frac{\Delta}{\rho^2}\left(dt - \frac{a_1 \sin^2\theta}{\Xi_1}d\phi_1 - \frac{a_2\cos^2\theta}{\Xi_2}d\phi_2\right)^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a_1 dt - \frac{(r^2+a_1^2)}{\Xi_1}d\phi_1\right)^2 + \\ &+\frac{\Delta_\theta\cos^2\theta}{\rho^2}\left(a_2 dt - \frac{(r^2+a_2^2)}{\Xi_2}d\phi_2\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \\ &+\frac{(1+r^2)}{r^2\rho^2}\left(a_1 a_2 dt - \frac{a_2(r^2+a_1^2)\sin^2\theta}{\Xi_1}d\phi_1 - \frac{a_1(r^2+a_2^2)\cos^2\theta}{\Xi_2}d\phi_2\right)^2, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \Delta &= \frac{1}{r^2}(r^2+a_1^2)(r^2+a_2^2)(1+r^2) - 2m\,, \\ \Delta_\theta &= 1 - a_1^2\cos^2\theta - a_2^2\sin^2\theta\,, \\ \rho^2 &= r^2 + a_1^2\cos^2\theta + a_2^2\sin^2\theta\,, \\ \Xi_i &= 1 - a_i^2\,. \end{aligned} \tag{2.2}$$

The parameter m is related to the black hole mass and a_i to the angular velocities.

The particular case $m = 0$ corresponds to empty AdS. The asymptotic AdS nature of the metric (2.1) can be exhibited by introducing new coordinates

$$\begin{aligned} t &= t\,, \\ \Xi_1 y^2 \sin^2\Theta &= (r^2 + a_1^2)\sin^2\theta\,, \\ \Xi_2 y^2 \cos^2\Theta &= (r^2 + a_2^2)\cos^2\theta\,, \\ \Phi_i &= \phi_i + a_i t\,. \end{aligned} \tag{2.3}$$

The horizon radius r_+ is defined as the largest root of $\Delta = 0$. In the coordinates (2.1) both the horizon and the sphere at infinity rotate. The angular velocities

$$\Omega_i = \Omega_i^H - \Omega_i^\infty = \frac{a_i(1 + r_+^2)}{r_+^2 + a_i^2}\,, \tag{2.4}$$

act as chemical potentials for the angular momenta of the fields in the dual field theory.

We have set the scale of AdS to one such that the period of the euclidean time coordinate is

$$\beta = \frac{1}{T} = \frac{4\pi(r_+^2 + a_1^2)(r_+^2 + a_2^2)}{r_+^2\Delta'(r_+)\,.} \tag{2.5}$$

With these conventions $\beta = 1/T$ and Ω_i are taken to be dimensionless. The euclidean action calculated with respect to AdS is

$$I = -\log Z = -\frac{\pi\beta(r_+^2 + a_1^2)(r_+^2 + a_2^2)(r_+^2 - 1)}{8G_5(1 - a_1^2)(1 - a_2^2)}\,. \tag{2.6}$$

G_5 is the Newton's constant in five dimensions. The thermodynamic potential is defined by $F = TI$. Similar as in the non-rotating case there is a phase transition [19] at $r_+ = 1$. The field theory interpretation is that there is a deconfining phase for $r_+ > 1$ [20]. This is also the regime where we want to compare with the field theory at weak coupling. In order to do so we would like to express the thermodynamic potential in terms of T and Ω_i. In the large T and large r_+ regime we can invert (2.4) and (2.5) approximately

$$\begin{aligned} r_+ &= \pi T - \frac{1 - \Omega_1^2 - \Omega_2^2}{2\pi T} + O\Big(\frac{1}{T^2}\Big)\,, \\ a_i &= \Omega_i\left(1 - \frac{1 - \Omega_i^2}{\pi^2 T^2}\right) + O\Big(\frac{1}{T^3}\Big)\,. \end{aligned} \tag{2.7}$$

Using these expressions we find for the thermodynamic potential

$$F = -\frac{VN^2T^4}{(1 - \Omega_1^2)(1 - \Omega_2^2)}\left[\frac{\pi^2}{8} - \frac{3}{8T^2}\left(1 - \frac{\Omega_1^2 + \Omega_2^2}{3}\right) + O\left(\frac{1}{T^4}\right)\right], \tag{2.8}$$

where $V = 2\pi^2$ is the volume of the unit S^3. We have used the AdS/CFT dictionary $G_5 = G_{10}/\mathrm{Vol}(S^5) = \pi/2N^2$.

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2.2 Summary of thermodynamics of four dimensional Kerr-AdS black holes

The four dimensional Kerr-AdS metric has first appeared in [21]. It was subsequently studied in [14] and [15]. We take our conventions from [14]

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - \frac{a}{\Xi}d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(adt - \frac{(r^2+a^2)}{\Xi}\right)^2, \quad (2.9)$$

with

$$\begin{aligned} \Delta &= (r^2+a^2)(1+r^2) - 2mr\,, \\ \Delta_\theta &= 1 - a^2\cos^2\theta\,, \\ \rho^2 &= r^2 + a^2\cos^2\theta\,, \\ \Xi &= 1 - a^2\,. \end{aligned} \quad (2.10)$$

The case $m = 0$ corresponds to empty AdS. Coordinates that exhibit the asymptotically AdS form are

$$\begin{aligned} t &= t\,, & y^2 &= \frac{r^2\Delta_\theta + a^2\sin^2\theta}{\Xi}\,, \\ y\cos\Theta &= r\cos\theta\,, & \Phi &= \phi - at\,. \end{aligned} \quad (2.11)$$

The horizon radius r_+ is the largest root of $\Delta = 0$. The angular velocity relevant to the dual field theory thermodynamics is

$$\Omega = \Omega^H - \Omega^\infty = \frac{a(1+r_+^2)}{r_+^2 + a^2}\,. \quad (2.12)$$

The period of the euclidean time coordinate is given by

$$\beta = \frac{1}{T} = \frac{4\pi(r_+^2 + a^2)}{\Delta'(r_+)}\,. \quad (2.13)$$

The euclidean action calculated with respect to the AdS background is

$$I = -\frac{\beta(r_+^2+a^2)(r_+^2-1)}{4G_4(1-a^2)r_+}\,, \quad (2.14)$$

which implies the existence of a phase transition at $r_+ = 1$. Again we are interested in the high T and large r_+ regime. The high temperature expansions of r_+ and Ω are given by

$$\begin{aligned} r_+ &= \frac{4\pi T}{3} - \frac{1-2\Omega^2}{4\pi T} + O\left(\frac{1}{T^2}\right), \\ a &= \Omega\left(1 - \frac{9(1-\Omega^2)}{16\pi^2T^2}\right) + O\left(\frac{1}{T^2}\right). \end{aligned} \quad (2.15)$$

Using this and the AdS/CFT dictionary for the M2-brane case, $G_4 = \frac{3}{2\sqrt{2}} N^{-3/2}$, we find the following expression for the thermodynamic potential $F = TI$

$$F = -\frac{8\sqrt{2}\, V N^{3/2}\, \pi^2\, T^3}{81\,(1-\Omega^2)} \left[1 - \frac{9\,(2-\Omega^2)}{16\pi^2 T^2} + O\left(\frac{1}{T^3}\right) \right] . \qquad (2.16)$$

Here $V = 4\pi$ is the volume of the unit two-sphere.

3. Field theory calculation

We want to calculate the thermodynamics of four-dimensional $\mathcal{N} = 4$ Yang-Mills on a rotating three-sphere in the weak coupling limit. At weak (zero) coupling we will calculate the 1-loop approximation to the partition function, in which colored degrees of freedom run in the loop. This is only consistent in the deconfined phase of the theory. The interacting $\mathcal{N} = 4$ theory on the sphere is in a deconfined phase only at high temperature. This holds even at weak coupling [20, 22]. Therefore we will concentrate in the high temperature regime.

The thermal part of the 1-loop contribution to the partition function of a field theory on S^3 where matter is forced to rotate with angular velocities Ω_1 and Ω_2, is

$$\log Z = -\sum_i \epsilon_i \sum_{l,m_1,m_2}^{\infty} \log\left(1 - \epsilon_i\, e^{-\beta(\omega_l^i + m_1\Omega_1 + m_2\Omega_2)}\right) , \qquad (3.1)$$

with i labelling the particle species, $\epsilon_i = 1$ for bosons and -1 for fermions, and ω_l^i is the energy of a mode of angular momentum l. We will assume that the scalars of $\mathcal{N} = 4$ are conformally coupled since supergravity determines the field theory metric only up to conformal transformations. Then the thermodynamics will depend only on the dimensionless quantities TR and $\Omega_i R$, where R is the radius of the sphere. The high temperature limit is equivalent to the large radius limit of the sphere keeping constant the speed of the boundary, $\Omega_i R$. Thus the leading term in the high temperature expansion of (3.1) can be obtained by substituting the discrete sums in (3.1) by integrals [14, 18]. We will be interested in obtaining subleading terms; for that we will have to deal directly with the sums in (3.1).

3.1 Approximate evaluation of sums

As a first step we would like to evaluate sums of the generic form

$$\sum_{l=l_1}^{l_2} f_l \, , \qquad (3.2)$$

where f_l depends of an small parameter ϵ and we can define a function $f(x)$ such that $f_l = \epsilon f(\epsilon l)$. In the limit of very small ϵ, we have formally $\epsilon \Delta l \to dx$ ($\Delta l = 1$

in (3.2)). Our only hypothesis will be that $F(x) = \int f(x)\,dx$ is analytic in the interval $[x_1, x_2]$, with $x_1 = \epsilon l_1$ and $x_2 = \epsilon(l_2 + 1)$. We have then

$$\int_{x_1}^{x_2} f(x)\; dx = \sum_{l=l_1}^{l_2} [F(\epsilon(l+1)) - F(\epsilon l)]\,. \tag{3.3}$$

We can expand the r.h.s. in a Taylor series, obtaining

$$\sum_{l=l_1}^{l_2} \epsilon f(\epsilon l) = \int_{x_1}^{x_2} f(x)\, dx - \sum_{k=1}^{\infty} \frac{\epsilon^{k+1}}{(k+1)!} \sum_{l=l_1}^{l_2} f^{(k)}(\epsilon l)\,, \tag{3.4}$$

where $f^{(k)} = d^k f/dx^k$. Applying recursively this reasoning to $f^{(k)}$ we can evaluate (3.2) to any order in the small parameter ϵ. Up to order ϵ^4 it is straightforward to obtain[3]

$$\begin{aligned}\sum_{l=l_1}^{l_2} \epsilon f(\epsilon l) = &\left(F(x_2) - \frac{\epsilon}{2} f(x_2) + \frac{\epsilon^2}{12} f'(x_2) - \frac{\epsilon^4}{720} f'''(x_2)\right) - \\ &- \left(F(x_1) - \frac{\epsilon}{2} f(x_1) + \frac{\epsilon^2}{12} f'(x_1) - \frac{\epsilon^4}{720} f'''(x_1)\right).\end{aligned} \tag{3.5}$$

This expression is not the final answer since we will consider situations in which both the functions that appear in (3.5) and x_1, x_2 can have an explicit dependence on ϵ. In that case we will have to further Taylor expand (3.5) to the desired order in ϵ.

As a test of this approach, we will first calculate the high temperature expansion of the energy of $\mathcal{N} = 4$ Yang-Mills on S^3 without rotation at weak coupling. This was obtained long ago in [11] and recently reobtained using the heat kernel method in [12]. The 1-loop contribution to the thermal part of the energy of $\mathcal{N} = 4$ Yang-Mills on S^3 is

$$E = \sum_{s=0,1/2,1} n_s \sum_{l=s}^{\infty} \frac{d_l^s \omega_l^s}{e^{\beta \omega_l^s} - (-)^{2s}}\,, \tag{3.6}$$

where s denotes the spin of scalars, fermions and gauge bosons, d_l^s is degeneracy of particles with spin s and energy ω_l^s and n_s are the number of fields of each spin present in the vector multiplet of $\mathcal{N} = 4$, i.e. $n_0 = 6$, $n_{1/2} = 4$ and $n_1 = 1$. The data for scalars and fermions are

$$\begin{aligned} s = 0 :&\quad d_l = (l+1)^2\,, \qquad l \geq 0\,, \\ s = \frac{1}{2} :&\quad d_l = 2\left(l + \frac{1}{2}\right)\left(l + \frac{3}{2}\right), \qquad l \geq \frac{1}{2}\,. \end{aligned} \tag{3.7}$$

In both cases $\omega_l = (l+1)/R$ and l runs on the integers for bosons and half-integer for fermions.

[3] For $\epsilon = 1$ this is the Euler-MacLaurin formula. We thank R. Emparan for pointing this out to us.

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We have to take care of a subtlety in the spectrum of gauge fields on the three-sphere at finite temperature. The spectrum of a gauge theory on S^3 in the Feynman gauge $\nabla^\mu A_\mu = 0$ has been found some time ago in [23]. The vector field can be classified according to the representation of the $\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ isometry of the three-sphere. We label its representations by (j_1, j_2). The temporal component is a scalar on the three-sphere and its modes fall into the $(l/2, l/2)$, $l \geq 0$ representation. The vector modes on the sphere form the

$((l-1)/2, (l+1)/2)$, $((l+1)/2, (l-1)/2)$ and $(l/2, l/2)$, $l \geq 1$ representations. We denote the $((l-1)/2, (l+1)/2)$ and $((l+1)/2, (l-1)/2)$ fields by $\vec{A}_\pm$ and the $(l/2, l/2)$ modes of the three-vector by $\vec{A}_0$. The energies are given by

$$\begin{aligned}
A_0 &: \quad \omega_l = \frac{\sqrt{l(l+2)}}{R}, \qquad l \geq 0\,, \\
\vec{A}_0 &: \quad \omega_l = \frac{\sqrt{l(l+2)}}{R}, \qquad l \geq 1\,, \\
\vec{A}_\pm &: \quad \omega_l = \frac{l+1}{R}, \qquad l \geq 1\,.
\end{aligned} \tag{3.8}$$

In addition we have the modes coming from the ghosts. They are (minimally coupled) scalars and therefore fall into the $(\frac{l}{2}, \frac{l}{2})$ representations.

$$c\,, \bar{c}\,:\; \omega_l = \frac{\sqrt{l(l+2)}}{R}\,, \qquad l \geq 0\,. \tag{3.9}$$

The ghost fields have to be taken periodic around the S^1 and thus are subject to the Bose-Einstein distribution despite their fermionic nature. They contribute with negative sign to (3.6). We see now that the ghosts compensate the A_0 and the $\vec{A}_0$ contributions up to a left over zero energy mode. If we were on $R \times S^3$ we could gauge away the zero mode of the temporal component of the gauge field and there would be no reason to include the zero modes of the ghosts. On $S^1 \times S^3$ however the gauge field zero mode can not be removed by a proper (periodic on S^1) gauge transformation. Therefore these zero modes have to be included in the spectrum. The contribution from the left over zero mode to (3.6) is

$$E = -\lim_{\omega \to 0} \frac{\omega}{e^{\beta\omega} - 1} = -\frac{1}{\beta}\,. \tag{3.10}$$

We have to add this to the contributions from the transversal gauge field modes.

After these remarks we can apply (3.5) to (3.6). The small parameter is $\epsilon = 1/TR$. The function $f_i(x)$ associated to each species can be obtained simply by substituting $l \to x/\epsilon$ in (3.6). The upper limit of the continuous variable is $x_2 = \infty$. All functions appearing in the r.h.s. of (3.5) tend to zero at infinity,[4] therefore the sum

[4]The function F is only defined up to a constant. This constant can be taken such that $F(x \to \infty) = 0$.

is determined by the value of the functions at x_1. We have $x_1 = 0, \epsilon/2, \epsilon$ for scalars, fermions and (the transversal modes of the) gauge bosons respectively, obtaining

$$E = -T^4 R^3 \left[\sum_{s=0,1/2,1} n_s \left(F_s(\epsilon s) - \frac{\epsilon}{2} f_s(\epsilon s) + \frac{\epsilon^2}{12} f_s'(\epsilon s) - \frac{\epsilon^4}{720} f_s'''(\epsilon s) \right) + O(\epsilon^5) \right], \tag{3.11}$$

where all functions are well defined in the limit $\epsilon \to 0$. We have now to Taylor expand each function around $\epsilon = 0$. The definition (3.1) involves logarithms which make Taylor expansions around $\epsilon = 0$ ill-defined. This is the reason why we chose to calculate the energy instead of $\log Z$. We can then use the relation $E = -\frac{\partial}{\partial \beta} \log Z$ to derive the partition function up to a temperature independent term. From (3.11) we obtain the contribution of each species to the energy of the gauge theory [11, 12]

$$\begin{aligned} s = 0 \; : & \quad 6V \left(\frac{\pi^2 T^4}{30} - \frac{1}{480 \pi^2 R^4} \right), \\ s = \frac{1}{2} \; : & \quad 4V \left(\frac{7\pi^2 T^4}{120} - \frac{T^2}{48 R^2} - \frac{17}{1920 \pi^2 R^4} \right), \\ s = 1 \; : & \quad V \left(\frac{\pi^2 T^4}{15} - \frac{T^2}{6R^2} + \frac{T}{2\pi^2 R^3} - \frac{11}{240 \pi^2 R^4} \right), \end{aligned} \tag{3.12}$$

where $V = 2\pi^2 R^3$ is the volume of S^3. The term at order $\sim T/R^3$ for the gauge bosons is precisely cancelled by the contribution of the left over ghost zero mode (3.10). The terms independent of the temperature equal minus the Casimir energy of each field on the sphere. The final result for the energy is

$$E = V \left(\frac{\pi^2 T^4}{2} - \frac{T^2}{4R^2} \right) - E_C + O\left(\frac{1}{TR^2} \right), \tag{3.13}$$

with E_C denoting the Casimir energy of $\mathcal{N} = 4$ Yang-Mills on a three-sphere.

3.2 $\mathcal{N} = 4$ Yang-Mills on a rotating 3-sphere

After having tested our method on a simple example, we want to apply it to $\mathcal{N} = 4$ Yang-Mills on a rotating three-sphere. The partition function can be used to obtain the thermodynamic potential $F = -T \log Z$ in the grand-canonical ensemble, where the thermodynamic variables are the temperature and the angular velocities. As before, we will calculate $\frac{\partial}{\partial \beta} \log Z$ instead of $\log Z$. We set in the following $R = 1$ for convenience. Therefore the small parameter will be just the inverse of the temperature, β. We will analyse separately the contribution of scalars,[5] fermions and gauge bosons, which in terms of the spin of each species is given by [16]

$$-\frac{\partial \log Z}{\partial \beta} = \sum_{l=s}^{\infty} \sum_{m=-\frac{l+s}{2}}^{\frac{l+s}{2}} \sum_{n=\frac{l-s}{-2}}^{\frac{l-s}{2}} \frac{l + 1 + m\Omega_+ + n\Omega_-}{e^{\beta(l+1+m\Omega_+ + n\Omega_-)} - (-)^{2s}} + (\Omega_+ \leftrightarrow \Omega_-). \tag{3.14}$$

[5] Expression (3.14) gives twice the contribution of a real scalar. Notice that for $s = 0$ the summation is already symmetric under the interchange of Ω_+ and Ω_-.

Here $\Omega_\pm$ are the angular velocities corresponding to the Cartan elements of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ rotation group of the three-sphere. The angular velocities (2.4) are related to them by $\Omega_\pm = \Omega_1 \pm \Omega_2$ [16].

We will only be interested in the leading and first subleading term of the high temperature expansion. For that it is enough to evaluate the sums to order β^2. We have to apply three times (3.5). The calculation simplifies by the fact that we can define a single variable x such that when $x = \beta(l+1) + \beta m\Omega_+ + \beta n\Omega_-$ the function

$$f(x) = \frac{x}{e^x - (-)^{2s}}\,, \tag{3.15}$$

equals βf_{lmn}, where f_{lmn} denote the summand in (3.14). The function $f(x)$ and its integrals are analytic between $[0,\infty]$. The result of the n summation is

$$\sum_n f_{lmn} = T^2 \sum_{i=1,2} (-)^i \left[\frac{1}{\Omega_+}F(x_i) - \frac{\beta}{2} f(x_i) + \frac{\beta^2}{12}\Omega_+ f'(x_i)\right], \tag{3.16}$$

where $x_2 = (\beta(l+1) + \beta m\Omega_+ + \beta(\frac{l-s}{2}+1)\Omega_-)$, $x_1 = (\beta(l+1) + \beta m\Omega_+ + \beta(-\frac{l-s}{2})\Omega_-)$, $F = \int f dx$ and $f' = df/dx$. Defining integrals and derivatives with respect to x instead of the continuous variable associated to n is the reason for the extra factors of Ω_+. It is equally straightforward to perform the summation in m, obtaining

$$\begin{aligned}\sum_m \sum_n f_{lmn} = T^3 \sum_{i,j=1,2} \frac{(-)^{i+j}}{\Omega_+\Omega_-} \Big[& h(x_{ij}) - \frac{\beta}{2}(\Omega_+ + \Omega_-)F(x_{ij}) + \\ & + \frac{\beta^2}{12}(\Omega_+^2 + 3\Omega_+\Omega_- + \Omega_-^2) f(x_{ij})\Big],\end{aligned} \tag{3.17}$$

where $h = \int F dx$ and the four values x_{ij} coming from the upper and lower bound of the two summations are

$$\begin{aligned} x_{ij} &= \beta\alpha_{ij} l + \beta\gamma_{ij}\,, \qquad i,j = 1,2\,, \\ \alpha_{ij} &= 1 + \frac{(-)^i\Omega_+ + (-)^j\Omega_-}{2}\,, \\ \gamma_{ij} &= 1 + \frac{1 + (-)^i(1+s)}{2}\Omega_+ + \frac{1+(-)^j(1-s)}{2}\Omega_-\,. \end{aligned} \tag{3.18}$$

Now we can perform the last summation. The functions $H = \int h dx$, h and F are defined only up to constants. We will choose these constant such that all of these functions vanish as $x \to \infty$. The initial function f (3.15) also vanishes at infinity. Therefore the triple sum is

$$\begin{aligned}\sum_l \sum_m \sum_n f_{lmn} = T^4 \sum_{i,j=1,2} \frac{(-)^{i+j}}{\Omega_+\Omega_-} \Big[& -\frac{1}{\alpha_{ij}} H(\beta\delta_{ij}) + \frac{\beta}{2}\Big(1 + \frac{\Omega_+ + \Omega_-}{\alpha_{ij}}\Big) h(\beta\delta_{ij}) - \\ & - \frac{\beta^2}{12}\Big(\alpha_{ij} + 3(\Omega_+ + \Omega_-) + \frac{\Omega_+^2 + 3\Omega_+\Omega_- + \Omega_-^2}{\alpha_{ij}}\Big) F(\beta\delta_{ij})\Big]\end{aligned} \tag{3.19}$$

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with $\delta_{ij} = \alpha_{ij} s + \gamma_{ij}$. The last step consists in expanding each function around $\beta = 0$ and finally adding the same expression with Ω_+ and Ω_- interchanged. The coefficient of the leading T^4 term is simply given by

$$-\frac{2H(0)}{\Omega_+\Omega_-} \sum_{i,j=1,2} \frac{(-)^{i+j}}{\alpha_{ij}} \,. \tag{3.20}$$

The factor 2 appears because (3.20) is symmetric under the interchange of Ω_+ and Ω_-. The next order contribution is

$$\frac{h(0)}{2\,\Omega_+\Omega_-} \sum_{i,j=1,2} (-)^{i+j} \left[\frac{-2\delta_{ij} + \Omega_+ + \Omega_-}{\alpha_{ij}} + (\Omega_+ \leftrightarrow \Omega_-) \right] . \tag{3.21}$$

Using (3.18) one can see that (3.21) is equal to zero. At the order T^2 we obtain the following expression

$$\begin{aligned} -\frac{F(0)}{12\Omega_+\Omega_-} \sum_{i,j=1,2} (-)^{i+j} \Bigg[& \frac{6\,\delta_{ij}^2}{\alpha_{ij}} - 6\delta_{ij}\left(1 + \frac{\Omega_+ + \Omega_-}{\alpha_{ij}}\right) + \\ & + \alpha_{ij} + \frac{\Omega_+^2 + 3\Omega_+\Omega_- + \Omega_-^2}{\alpha_{ij}} + (\Omega_+ \leftrightarrow \Omega_-) \Bigg] . \end{aligned} \tag{3.22}$$

Integrating (3.14) we obtain the contribution to the thermodynamical potential of real scalars, Weyl fermions and gauge bosons living on a rotating 3-sphere

$$\begin{aligned} F_{s=0} &= \frac{V}{(1-\Omega_1^2)(1-\Omega_2^2)} \left[-\frac{\pi^2 T^4}{90} + \frac{T^2}{72}(\Omega_1^2 + \Omega_2^2) \right] , \\ F_{s=1/2} &= \frac{V}{(1-\Omega_1^2)(1-\Omega_2^2)} \left[-\frac{7\pi^2 T^4}{360} + \frac{T^2}{48}\left(1 - \frac{\Omega_1^2 + \Omega_1^2}{3}\right) \right] , \\ F_{s=1} &= \frac{V}{(1-\Omega_1^2)(1-\Omega_2^2)} \left[-\frac{\pi^2 T^4}{45} + \frac{T^2}{6}\left(1 - \frac{5(\Omega_1^2 + \Omega_2^2)}{6}\right) \right] . \end{aligned} \tag{3.23}$$

We have used $\Omega_\pm = \Omega_1 \pm \Omega_2$. For the particular case of $\mathcal{N} = 4$ Yang-Mills we have

$$F_{\mathcal{N}=4} = \frac{VN^2}{(1-\Omega_1^2)(1-\Omega_2^2)} \left[-\frac{\pi^2 T^4}{6} + \frac{T^2}{4}\left(1 - \frac{\Omega_1^2 + \Omega_2^2}{3}\right) \right] . \tag{3.24}$$

In [18] the leading contribution to the thermodynamics of $\mathcal{N} = 4$ Yang-Mills on a rotating S^3 was analysed in the high temperature limit. The discrepancy by a factor 3/4 between the strong and weak coupling limits of the thermodynamics potentials which holds for the gauge theory in flat non-rotating space, was found in [18] to hold exactly also for the case with one rotation parameter, i.e. $\Omega_2 = 0$. This is a somehow surprising result since the rotations introduce new dimensionless parameters such that strong and weak coupling limits could differ by an arbitrary function of the Ω's. Comparing (2.8) and (3.24) we observe that the simple factor

3/4 between the T^4 term at strong and weak coupling persists in the generic case with two rotations. Moreover, a stronger result holds. The strong and weak coupling contribution to the thermodynamic potentials at the subleading order T^2 differ again only by a numerical factor, 3/2. There is no obvious symmetry reason that could explain this behaviour.

The relation between strong and weak coupling limits of the thermodynamic potential for the system with two rotations as a function of the horizon radius r_+ was investigated numerically in [16]. The ratio F_{st}/F_w decreases from 3/4 at $T \to \infty$ to zero at the Hawking-Page phase transition point $r_+ = 1$. However it decreases slower for systems with stronger rotation. We can reproduce that result for high temperature from (3.24)

$$\frac{F_{st}}{F_w} = \frac{3}{4}\left[1 - \frac{3\beta^2}{2\pi^2}\left(1 - \frac{\Omega_1^2 + \Omega_2^2}{3}\right)\right]. \tag{3.25}$$

The term in parenthesis varies between 1 in the non-rotating case and 1/3 when both $\Omega_1, \Omega_2 \to 1$.

3.3 The $\mathcal{N}=8$ supersingleton on a rotating sphere

As already mentioned in the introduction, the four dimensional Kerr-AdS black hole is thought to be dual to the superconformal field theory arising on coinciding M2-branes. We only have a lagrangean formulation for the case of the field theory on a single M2-brane. This is the $\mathcal{N} = 8$ supersingleton theory consisting of eight conformally coupled scalars and eight spinors. It seems most natural to compare the large N limit of this theory represented by supergravity to the $N = 1$ field theory, since there is no dimensionless coupling as in the four dimensional case. For the same reason as before we will actually compute $\frac{\partial}{\partial\beta}\log Z$

$$-\frac{\partial}{\partial\beta}\log Z = \sum_{l=s}^{\infty}\sum_{m=-l}^{l}\frac{l + 1/2 + m\Omega}{e^{\beta(l+1/2+m\Omega)} - (-)^{2s}} \tag{3.26}$$

The calculation parallels the previous case. Therefore we do not present the details. We obtain for the thermodynamic potential

$$F = -\frac{V\,T^3}{1 - \Omega^2}\left[\frac{7\zeta(3)}{\pi} + \frac{1 - 2\Omega^2}{6\pi}\frac{\log(T)}{T^2}\right]. \tag{3.27}$$

It has been noted in [24] that the leading term contains a ζ-function with odd integer argument. For this reason there can not be an agreement up to some rational number between the supergravity result and the field theory result. For the leading term there is still a minimal agreement between the two results since their scaling with the temperature is the same. This is however to be expected because the leading term corresponds to the flat space limit and in this case conformal symmetry dictates the scaling with T. The scaling of the subleading term can however not be predicted by conformal symmetry. As we see now it is indeed different. The field theory calculation shows logarithmic behaviour, whereas the supergravity result gives polynomial behaviour.[6] In fact even the functional dependence on Ω is different for the subleading terms. In the light of these results it seems even more remarkable that the four dimensional calculations show such a high agreement.

Acknowledgments

We would like to thank E. Alvarez and M. Tytgat for discussions.

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